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A Characterization of the Zero-One Inflated Logarithmic Series Distribution

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***Abstract*—In this paper, a characterization of zero-one inflated logarithmic series distributions through a linear differential equation of its probability generating function is given.**

Keywords-Logarithmic Series Distribution, Zero-One Inflated Logarithmic Series Distribution, Probability Generating Function, Linear Differential Equation.

I. INTRODUCTION

Logarithmic series distribution (LSD) sometime called logarithmic distribution or log-series distribution. This distribution is a member of the class of generalized power series distributions. Its zero-inflated form was studied by many researchers recently. In particular, Kumar and Riyaz [1] developed an extended version of the zero-inflated logarithmic series distribution (ZILSD) and derive some of its structural and estimating its parameters, then, they [2] considered a modified form of the ZILSD and studied some of its properties with the maximum likelihood estimator of its parameters applied to real life data sets. Later, they [3,4,5] developed an order k version of the ZILSD and considered some of its structural properties, considered its modified version, study some of its properties, and proposed an alternative version of the ZILSD and studied some of its properties with some applications. Alshkaki [6] characterized the ZILSD through a linear differential equation.

In this paper, we introduce in Section 2, the definition of the LSD and its zero-one inflated form with their probability generating function (pgf), followed in Section 3 by a characterize the ZOILSD through a linear equation of its pgf.

II. ZERO-ONE INFLATED LOGARITHMIC SERIES DISTRIBUTIONS

Let $\theta \in (0,1)$, then the discrete random variable (rv) X having probability mass function (pmf);

$$\mathrm{P}(X=x)=\begin{cases}\frac{\theta^x}{-x\log(1-\theta)}, & x=1,2,3,\\ 0 & otherwise,\end{cases} \quad (1)$$

is said to have a LSD with parameter θ. We will denote that by writing $X \sim LSD(\theta)$. For a detailed historical remarks and genesis of the LSD see Johnson et al [7], pp 303-305, and pp 305-325 for further details about the LSD.

Let $X \sim LSD(\theta\ )$ as given in (1), let $\alpha \in (0,1)$ be a proportion of zero added to the rv X, and let $\beta \in (0,1)$ be an extra proportion added to the proportion of ones of the rv X, such that $0 < \alpha + \beta < 1$, then the rv Z defined by;

$$P(Z=z)=\begin{cases}\alpha, & z=0\\ \beta+(1-\alpha-\beta)\dfrac{\theta}{-\log(1-\theta)}, & z=1\\ (1-\alpha-\beta)\dfrac{\theta^z}{-z\log(1-\theta)}, & z=2,3,4,\ldots\\ 0 & otherwise,\end{cases} \quad (2)$$

is said to have a zero-one inflated logarithmic series distribution (ZOILSD), and we will denote that by writing $Z \sim ZOILSD(\theta;\alpha,\beta)$.

Note that, if $\beta \to 0$, then (2) reduces to the form of the ZILSD. Similarly, the case with $\alpha \to 0$ and $\beta \to 0$, reduces to the standard case of LSD.

The pgf of the rv X, $G_X(t)$, is given by;

$$G_X(t)=E(t^X)$$

$$=\frac{1}{-\log(1-\theta)}\sum_{x=1}^{\infty}\frac{\theta^x}{x}t^x$$

$$=\frac{\log(1-\theta t)}{\log(1-\theta)}, \quad (3)$$

while the pgf of the rv *Z*, can be shown to be;

$$G_Z(t) = \alpha + \beta t + (1-\alpha-\beta)\frac{log(1-\theta t)}{log(1-\theta)} \qquad (4)$$

III. CHARACTERIZATION OF THE ZERO-ONE INFLATED LOGARITHMIC SERIES DISTRIBUTION

We give below the main result of this paper.

Theorem: The discrete rv Z taking non-negative integer values, has a ZOILSD if its pgf, $G(t)$ satisfies for some non-zeros numbers *a*, *b*, *c* and *d*, that;

$$(a+bt)\frac{\partial}{\partial t}G(t) = c + dt \qquad (5)$$

Proof: Without loss of generality, let us assume that $a = 1$, hence (5) can be written as, with *d* replaced by *h* in order not to be confused with derivatives symbol *d*;

$$\frac{\partial}{\partial t}G(t) = \frac{c+ht}{1+bt}$$

hence,

$$G(t) = \int \frac{c+ht}{1+bt}dt$$

$$= \int \frac{c}{1+bt}dt + h\int \frac{t}{1+bt}dt \qquad (6)$$

In the second integral, let *u* = $1 + bt$, then *du* = *bdt*, therefore after subtitling that in (6) in the second integral, and evaluating both integrals, we simply get that;

$$G(t) = \frac{(bc-d)}{b^2}log(1+bt) + \frac{d}{b}t + k$$

where *k* is a an arbitrary constant.

Since *1* = $G(1)$;

$$1 = \frac{(bc-d)}{b^2}log(1+b) + \frac{d}{b} + k,$$

and hence;

$$k = 1 - \frac{d}{b} - \frac{(bc-d)}{b^2}log(1+b),$$

therefore;

$$G(t) = \frac{(bc-d)}{b^2}log(1+bt) + \frac{d}{b}t + 1 - \frac{(bc-d)}{b^2}log(1+b) - \frac{d}{b},$$

or equivalently,

$$G(t) = 1 - A + At + B\,log(1+bt) - B\,log(1+b), \qquad (7)$$

where;

$$A = \frac{d}{b} \qquad (8)$$

$$B = \frac{(bc-d)}{b^2} \qquad (9)$$

Using (7), we have that;

$$P(Z=0) = G(0) = 1 - A - B\,log(1+b)$$

Consider now, the derivatives of $G(t)$. We have that;

$$\frac{\partial}{\partial t}G(t) = A + B\,\frac{b}{1+bt}$$

and hence;

$$P(Z=1) = \frac{\partial}{\partial t}G(0) = A + B\,b$$

Also;

$$\frac{\partial^{(2)}}{\partial t^2}G(t) = -B\,\frac{b^2}{(1+bt)^2}$$

In general;

$$\frac{\partial^{(n)}}{\partial t^n}G(t) = (n-1)!\,B\frac{(-1)^{n-1}b^n}{(1+bt)^n}, \qquad n = 2, 3, \ldots$$

It follows that;

$$P(Z=z) = \frac{1}{z!}\frac{\partial^{(z)}}{\partial t^z}G(0)$$

$$= \frac{B}{z}(-1)^{z-1}b^z, \qquad z = 2, 3, \ldots,$$

therefore;

$$P(Z=z)=\begin{cases} 1-A-B\,log(1+b), & z=0 \\ A+Bb, & z=1 \\ \frac{B}{z}(-1)^{z-1}b^z, & z=2,3,4,\dots \\ 0, & otherwise, \end{cases} \quad (10)$$

Let;

$$\theta=-b \quad (11)$$

$$\alpha=\ 1-A-B\,log(1+b)$$

$$\beta=\ A,$$

then,

$$\alpha\ =\ 1-\frac{d}{b}-\frac{(bc-d)}{b^2}log(1+b) \quad (12)$$

$$\beta=\frac{d}{b} \quad (13)$$

and therefore;

$$B=(1-\alpha-\beta)\frac{1}{log(1+b)},$$

hence, the pmf of the rv Z, $P(Z=z)$, given in (10) can be written as the form given in (2), with $\theta,\ \alpha$ and β, given by (11), (12) and (13), respectively.

Let us consider possible values of $\theta,\ \alpha$ and β, given by (11), (12) and (13), respectively. Now if; $-1<b<0$, then θ given by (11) satisfies that; $0<\theta<1$. If $-1<b<d<0$, the β given by (13) satisfies that $0<\beta<1$. If c in (12) is given by; $\frac{d}{b}<c<\frac{d-1}{b}$, then $0<-(bc-d)<1$, and since $-1<b<0$ and $0<-log(1-\theta)<1$; it follows that $0<\frac{(bc-d)}{b^2}log(1+b)<1$, or equivalently; $0<1-\alpha-\beta<1$, and thus $0<\alpha+\beta<1$, implying that $0<\alpha<1$ since $0<\beta<1$. This completes the proof.

Theorem 2: Let Z be a discrete rv taking non-negative integer values, then $Z\sim ZOILSD(\theta;\alpha,\beta)$, for some non-zero θ,β and α if and only if its pgf satisfying (5) for some non-zero numbers *a*, *b*, *c* and *d*.

Proof: if $Z\ \sim ZOILSD(\theta;\alpha,\beta)$ for some non-zero θ,α and β, then it easy 5.1) with $a\ =\ 1$, $b\ =-\theta$ and $c\ =\beta-(1-\alpha-\beta)\frac{\theta}{log(1-\theta)}$ and d = - $\beta\theta$, and hence the proof is complete using Theorem 1.

Theorem 2 leads to the following conclusion obtained by Alshkaki [6] that characterize the ZILSD.

Theorem 3: Let Z be a discrete rv taking non-negative integer values, then $Z\sim ZILSD(\theta;\alpha)$, for some non-zero θ and α if and only if its pgf satisfying that;

$$(a+bt)\frac{\partial}{\partial t}G(t)=c$$

for some non-zero numbers *a*, *b* and *c*.

Proof: Just let $d\rightarrow 0$ in Theorem 2.

Conclusions

We introduced a characterization of the zero-one inflated logarithmic series distributions through a linear differential equation of its probability generating function. We would propose an extension of this results to other forms and distributions.

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